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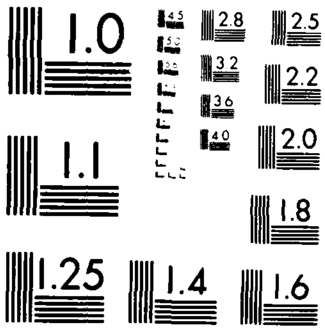
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AXIAL REPRESENTATIONS OF SHAPE

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ABSTRACT

Classes of "ribbonlike" planar shapes can be defined by specifying an arc, called the spine or axis, and a geometric figure such as a disk or line segment, called the generator, that "sweeps out" the shape by moving along the spine, changing size as it moves. Shape descriptions of this type have been considered by Blum, Brooks, Brady, and others. This paper considers such descriptions from the standpoints of both generation and recovery (i.e. given a shape generated in this way, to determine the axis and generation rule that gave rise to it), and discusses their relative advantages and disadvantages.

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1. Introduction

A number of authors have considered methods of describing ribbonlike planar shapes in terms of an arc called the "axis" or "spine" about which the shape is locally symmetric. This paper considers a number of such shape descriptions from the standpoints of both generation (of the shape, given the axis) and recovery (of the axis, given the shape), and discusses their relative advantages and disadvantages.

From the generative standpoint, we are given the spine and a geometric figure such as a disk or line segment, called the generator, that "sweeps out" the shape by moving along the spine, possibly changing size as it moves. More precisely, we assume that the generator contains a unique reference point - e.g., the center of the disk or the midpoint of the line segment. At each point P of the spine we place a copy of the generator so that its reference point coincides with P. The union of all these copies, which may be of different sizes, is the generated shape.

An early use of this approach to define shapes, due to Blum [1,2], used a disk as generator. Blum called this representation the "medial (or symmetric) axis transformation". Blum was more interested in description (i.e., recovery) than in generation; not surprisingly, it turns out that Blum descriptions are uniquely recoverable. In fact, Blum's method was developed to describe arbitrary shapes that are not necessarily ribbonlike, using spines (or "skeletons") that are not necessarily simple arcs; but in this paper we shall consider only the case where the spine is a simple arc.

More recently, Brooks [3] defined a class of shapes called "generalized ribbons", using a line segment as generator and requiring it to make a fixed angle with the spine. [Generalized ribbons are two-dimensional versions of "generalized cylinders" (sometimes called "generalized cones"), which were introduced in the early 1970's by Binford [4] and his students. In a generalized cylinder, the spine is a space curve and the generator is a planar figure that moves along the spine, making a constant angle with it and changing size as it moves, to sweep out a three-dimensional shape.] Brooks' definition is more flexible than Blum's from a generative standpoint, but as we shall see, it does not allow unique recovery.

Still more recently, Brady [5] introduced a shape representation based on "local symmetry". Here the "generator" is also a line segment, but it is required to make equal angles with the sides of the shape, rather than making a fixed angle with the spine. Brady's primary interest was in description, not in generation; not surprisingly, it turns out that his descriptions are usually recoverable, whereas generation is not straightforward.

We shall refer to these methods of defining or describing shapes as "axial representations", since they all involve a spine or axis which is a planar arc. Our interest in this paper is in the use of these methods to define "ribbonlike" shapes, and we shall consider various ways of restricting them so that they do indeed tend to define such shapes.

However, we make no claim that these classes of shapes coincide perceptually with the class of ribbonlike shapes. For brevity, we shall call the classes of shapes defined by these methods "Blum ribbons", "Brooks ribbons", and "Brady ribbons", respectively. We shall discuss these classes of shapes from the standpoint of generation as well as of recovery, even though their inventors may not have had generation in mind.

2. Some general observations

Before considering specific classes of axial representations, we make a few general observations about them, and introduce some general terminology and notation.

Figure 1 shows a piece of spine (labeled S) and one position of the generator. We shall usually assume that S is a simple, rectifiable arc with a tangent at every point, and that the generator is a simply connected set. The reference point of the generator, labeled O in the figure, will be called its <u>center</u>, and the generator whose center is at position O on S will be denoted by G_O. Let R (for "ribbon") be the union of all the G_O's for all $O \in S$. Since S and the G's are connected, so is R. (One can get from any point P of any G_{O_1} to any point Q of any G_{O_2} by moving within G_{O_1} from P to O_1, then along S to O_2, then within G_{O_2} from O_2 to Q.)

The G's are all geometrically similar figures; they all have the same shape, and differ only in size. We shall measure the size of G_O by its semidiameter (or "radius") r_O. We shall usually assume that r_O, as well as the orientation of G_O, vary continuously (and differentiably) as O moves along S.

Since we want R to look ribbonlike, it is reasonable to require that as G_O moves along S, it should not intersect G's located in other parts of S. If such intersections were indiscriminately allowed, we could "paint out" shapes that were very nonribbonlike, as illustrated in Figure 2. However, it is not always obvious how to define this concept of nonselfintersection, since when O_1 is close to O_2, G_{O_1} and G_{O_2} may have to intersect. We shall consider this issue further when we discuss specific classes of representations.

A closely related requirement, based on our desire that R look ribbonlike, is that the G's should not contain one another. By requiring this, we insure that the G's centered at every point of S have some influence on the appearance of R, so that the shape of R is generally "similar" to that of S (except that R is thick). Figure 3 shows examples of what could happen if we did not impose this restriction. We shall, in fact, impose the stronger requirement that the G_O's are all <u>maximal</u> -- in other words, no G_O is strictly contained in any G-shaped region that is contained in R. It follows that every G_O contains at least two border points of R; if it did not, we could expand it slightly to obtain a larger G-shaped region still contained in R, contradicting the maximality of G_O. Conversely, note that every border point of R is in some G_O since R is the union of the G_O's.

Let O' and O" be the endpoints of S. Those parts of the border of R that are in $G_{O'}$ or $G_{O''}$, but not in any other G_O, will be called the <u>ends</u> of R. The remaining parts of the border of R will be called the <u>sides</u> of R. These concepts are illustrated in Figure 4. Since S is smooth (i.e., differentiable), and since the size and orientation of G_O vary smoothly as O moves along S, it is not hard to see that the border of R must also be smooth. We shall denote the border of R by b_R.

We have not specified here that G be symmetric about its center O, but in the examples we shall consider this will in fact be the case (G will be a disk and O its center, or G will be a line segment and O its midpoint). Symmetry of G tends to make R "locally symmetric," but it does not imply any type of global symmetry, since the spine may be curved, and the orientation of G relative to the spine may vary.

3. Ribbons generated by disks ("Blum ribbons")

In Blum's medial axis representation, the generator is a disk with its center on the spine. We shall call a shape generated by Blum's method a "Blum ribbon."

Proposition 3.1. Let R be simply connected and let b_R be smooth; then any maximal disk D contained in R is tangent to b_R.

Proof: D must touch b_R, say at P. If D were not tangent to b_R at P, it would cross b_R and so contain points not in R, contradiction. ||

Proposition 3.1 holds not just for disks, but for any class of shapes that have smooth borders.

Proposition 3.2. If R is a Blum ribbon, every maximal disk D contained in R is one of the G's (and in particular, has its center on S); thus the set of maximal disks is the same as the set of G's.

Proof: Let D be tangent to b_R at P. As pointed out in Section 2, P must be contained in some G_O, and any G_O is maximal; hence by Proposition 3.1 G_O too is tangent to b_R at P. Thus, D and G_O are both tangent to b_R at P, and since both are maximal, they must be identical. ||

Theorem 3.3. If R is a Blum ribbon, the spine and set of generators of R are uniquely determined.

Proof: Given R, for every $P \in b_R$ we can construct the set of disks tangent to b_R and contained in R. Let D_P be the largest of these disks, so that D_P is a maximal disk. By Proposition 3.1, $\{D_P | P \in b_R\}$ are all of the maximal disks. By Proposition 3.2, this is the same as the set of generators G_O and the spine is the locus of their centers. ||

We thus see that Blum ribbons are very well behaved from the standpoint of recoverability. However, they are more limited or harder to deal with in other respects. A serious limitation is that a thick Blum ribbon cannot have points of high positive (=convex) curvature on its border. For example, the shape shown in Figure 5a cannot be a Blum ribbon; the set of centers of its maximal disks is evidently not a simple arc. [On the other hand, points of high negative curvature are allowable, as Figure 5b shows.]

There are several ways to define non-self intersection for Blum ribbons. One approach is to require that the sides don't intersect themselves (or each other). To define this concept more precisely, note that by the proof of Theorem 3.3, there is a one-to-one correspondence between the points P on a side of R and the points O on the spine; thus if the side intersects itself, two different O's will correspond to the same P. From a purely generative standpoint, it would be more appropriate to define non-self intersection in terms of the generators themselves; but it isn't obvious how to do this. For ribbons generated by straight line segments, we could simply require that no two generators intersect; but we cannot require this in the case of disk generators, since disks having centers sufficiently close together on the spine must intersect as long as their radii are bounded away from zero. Another possibility would be to require that whenever two generators intersect, their intersection contains part of the spine; but this doesn't work either, since as Figure 6 shows, when the axis is curved, two generators that just touch do not touch on the spine. A better definition seems to be the following: Let S be the set of centers of all the generators that intersect any given generator G; then S is an arc. This rules out cases like those shown in Figures 7 and 8. (Figure 7 is ruled out in any case because the generators are not all maximal disks; but Figure 8 is not ruled out by nonmaximality.) Note that Figure 7 also shows that thick Blum ribbons are limited in the rate at which they can turn. (A concave side of a Blum ribbon can turn rapidly, as we saw in Figure 5b; an even simpler example is that of a thick annulus with a very small inner radius.)

Another approach to handling non-self-intersection would be to regard the generator not as a disk, but as a pair of radii terminating at the points where the disk is tangent to the sides, as in the proof of Theorem 3.3, and to require that no two of these radius pairs intersect. Unfortunately, it's not obvious how to define a generation process purely in terms of radius pairs. The radius pair approach is more interesting from the standpoint of recovery [6]: If P and P' are corresponding points on the two sides, the normals at P and P' will intersect each other at a point O equidistant from P and P', and this will be the first time that either of these normals meets a normal that has not itself previously met any other normal. In this case, O is the spine point corresponding to P and P'.

4. Ribbons generated by line segments

Suppose next that the generator is a line segment with its midpoint on the spine. In general, we can allow both the length and the orientation of the segment (relative to the spine) to vary as it moves along the spine. We shall call a shape R generated in this way an L-ribbon. Note that by maximality, the endpoints of any generator must both lie on the border of R. In fact, the sides of R are just the loci of the two endpoints, while the ends of R are just the generators at the two ends of the spine.

It is trivial to formulate the non-selfintersection constraint for L-ribbons; we simply require that no two generators intersect. Thick L-ribbons are also not strongly limited in their ability to turn, thanks to the fact that the slope of the generator relative to the spine is allowed to vary; see Figure 9.

L-ribbons can also have points of high positive (or negative) curvature on their borders; thus they are also more flexible in this respect than Blum ribbons. In fact, an L-ribbon can have long protuberances on its border (Figures 10a-b), as long as every point on the protuberance is visible from the spine (Figures 10c-d). Since the slope of the generator is allowed to vary as it moves along the spine, an L-ribbon can even have protuberances with overhangs (Figure 10b). On the other hand, some combinations of protuberances may be impossible even if they are all visible from the spine, if the generators would have to cross one another in order to generate them (Figure 10e).

We see from Figure 10 that the class of L-ribbons is somewhat too large; it contains shapes that no longer look like "simple" ribbons. Some of these examples will be ruled out by the restrictions that we will impose in the next two sections; but the most basic example, Figure 10a, is not ruled out.

What seems to be the problem with Figure 10a is that when we regard a shape as being axially generated, we prefer to choose the spine so as to make the shape as elongated as possible, i.e., so as to maximize the length of the spine relative to the thickness of the shape. For example, a rectangle can be generated as an L-ribbon (with generator perpendicular to the spine) in two ways, as shown in Figures 11a-b, but we strongly prefer Figure 11a because it has greater elongatedness. Similarly, we prefer not to regard the protuberances in Figure 10 as being generated from the spine of the main part of the shape, even when this can be done legally, because the protuberances have much greater elongatedness with respect to spines of their own (Figure 12).

Another serious difficulty with L-ribbons is that they are highly ambiguous; the same shape can be generated in many different ways, even using the same axis, as illustrated in Figure 13. Of course, we strongly prefer the generation process in Figure 13a over that in Figure 13b because the former is much "simpler"; in Figure 13a both the size and slope of the generator remain constant, while in Figure 13b they both vary. For arbitrary shapes, however, it would not be easy to formulate a canonical measure of simplicity; for example, which would be preferred -- constant size and variable slope, or constant slope and variable size? The tradeoff between simplicity and degree of elongatedness is also far from clear.

Figures 11 and 13 show that L-ribbons are not recoverable; given the ribbon, the spine and set of generators are far from uniquely determined. Even if we had reliable criteria, based on simplicity and elongatedness, for preferring one generation process over another, we would still not have a constructive method of determining the best generation process for a given ribbon.

We can greatly reduce, or even eliminate, the ambiguity of L-ribbons if we require them to satisfy additional constraints. In the next two sections we consider two such constraints: (1) requiring the generators to make a fixed angle with the spine, and (2) requiring them to make equal angles with the sides of the ribbon.

5. The fixed angle case ("Brooks ribbons")

Our first restriction on L-ribbons is that the generator is required to make a fixed angle with the spine. We shall call this class of L-ribbons "Brooks ribbons." We shall assume here, for simplicity, that the angle between the generators and the spine is always 90°.

Fixing the angle has the undesirable consequence of limiting the ability of Brooks ribbons to make sharp turns. In fact, as Figure 14 shows, the thickness of a Brooks ribbon cannot exceed twice the radius of curvature of its spine. (On turning limitations for generalized cylinders see [7].) Brooks ribbons also still allow some pathological cases, such as that in Figure 10a.

A shape can be globally ambiguous with respect to Brooks ribbon generation, as we saw in Figure 11. However, the ambiguity in Figure 11 results from interchanging the roles of the end and the sides. If we specify which are the sides, Figure 11 becomes unambiguous. Indeed, we have

Proposition 5.1. If the sides of the Brooks ribbon R are straight and parallel, its spine and generators are uniquely determined; in fact, the spine is the line parallel to the sides and midway between them.

Proof: Let G_O be any generator, as illustrated in Figure 15. Since O is the midpoint of G_O, and the sides are parallel, by similar triangles O is midway between the sides. Since this is true for any O, the spine must be the line parallel to the sides and midway between them, and the generators must thus be perpendicular to the spine and the sides. ||

Note that by Proposition 5.1, a parallelogram cannot be a Brooks ribbon (in our restricted sense) unless it is a rectangle; since the generator must be perpendicular to the sides, it cannot generate the oblique ends (Figure 16). To generate oblique parallelograms, we must allow the generator to make an oblique angle with the spine. In what follows we shall ignore what happens at the ends of a ribbon, and consider only the problem of generating the parts of the sides away from the ends.

Let us now consider the case where the sides are straight but not parallel. Evidently, we can generate (parts of) these sides by taking the spine to be (part of) the straight line that bisects the angle between the sides, as shown in Figure 17. Surprisingly, however, this is not the only possibility. In fact, as we shall next prove, a Brooks ribbon with straight sides need not have a straight spine. It follows that Brooks ribbons are (locally) ambiguous; specifying (pieces of) the sides does not determine the spine.

In order to prove these assertions, consider first the general case where the sides are arbitrary curves $y=f(x)$ and $y=g(x)$, as shown in Figure 18. Let the (unknown) equation of the spine be $y=h(x)$. Let the generator centered at point $(x_0,h(x_0))$ of the spine hit the sides at pionts $(x_1,f(x_1))$ and $(x_2,g(x_2))$. Since the midpoint of the generator is on the spine, we must have $x_0=\frac{x_1+x_2}{2}$ and $h(x_0)=\frac{f(x_1)+g(x_2)}{2}$. The slope of the generator at $(x_0,h(x_0))$ is $-\frac{1}{h'(x_0)}$; thus its equation is $\frac{x_0-x}{y-h(x_0)} = h'(x_0)$, so that the intersection points with the sides satisfy

$$f(x_1) = \frac{x_0 - x_1}{h'(x_0)} + h(x_0) \qquad \text{and} \qquad g(x_2) = \frac{x_0 - x_2}{h'(x_0)} + h(x_0).$$

We can (in principle) solve these equations for x_1 and x_2 in terms of x_0, $h(x_0)$, and $h'(x_0)$, and substitute the results in the equation $x_0 = \frac{x_1+x_2}{2}$ to obtain an equation involving only x_0, $h(x_0)$, and $h'(x_0)$, i.e., a first order differential equation for the unknown function h.

As an example, let the sides be (pieces of) straight lines, say with equations $y=0$ and $y=mx$. Thus the intersection points satisfy

$$0 = \frac{x_0 - x_1}{h'(x_0)} + h(x_0) \qquad \text{and} \qquad mx_2 = \frac{x_0 - x_2}{h'(x_0)} + h(x_0)$$

This yields $$x_1 = x_0 + h(x_0)h'(x_0)$$

and $$x_2(mh'(x_0)+1) = x_0 + h(x_0)h'(x_0)$$

Thus $$x_0 = \frac{x_1+x_2}{2} = \tfrac{1}{2}[x_0+h(x_0)h'(x_0)][1+\frac{1}{1+mh'(x_0)}]$$

or $$2(1+mh'(x_0))x_0 = [x_0+h(x_0)h'(x_0)][2+mh'(x_0)]$$

which simplifies to

$$mx_0h'(x_0) = (2+mh'(x_0))h(x_0)h'(x_0)$$

Cancelling $h'(x_0)$ (clearly h is not a constant, so h' is not identically zero) gives

$$mh(x_0)h'(x_0)+2h(x_0)-mx_0 = 0$$

so that h satisfies the differential equation

$$myy'+2y-mx = 0$$

The general solution to this equation is found as follows:*
Let $y=xw$; then the equation becomes (cancelling x) $mw(w+xw')+2w-m=0$. Thus $xww'=\frac{1}{m}[m-2w-mw^2]=[1-\frac{2w}{m}-w^2]$, or $\frac{1}{x}+\frac{ww'}{w^2+\frac{2w}{m}-1}=0$.
It can be verified that

$$\frac{ww'}{w^2+\frac{2w}{m}-1} = \frac{aw'}{w+c} + \frac{bw'}{w+d}$$

where $a = \frac{\sqrt{m^2+1}+1}{2\sqrt{m^2+1}}$, $b = \frac{\sqrt{m^2+1}-1}{2\sqrt{m^2+1}}$, $c = \frac{1+\sqrt{m^2+1}}{m}$, and $d = \frac{1-\sqrt{m^2+1}}{m}$

Hence $\int\frac{dx}{x} + \int\frac{aw'dw}{w+c} + \int\frac{bw'dw}{w+d} = K$

or $\quad \log x + a\log(w+c) + b\log(w+d) = K$

or $\quad x(w+c)^a(w+d)^b = K'$.

Since $w=y/x$ and $a+b=1$, this becomes

$$(y+cx)^a(y+dx)^b = K'.$$

If we raise both sides to the power $2\sqrt{m^2+1}/m$, we get

$$(y+cx)^c(y+dx)^{-d} = K''$$

Noting finally that $-d=1/c$, we have

$$(y+cx)^c(y-\frac{1}{c}x)^{1/c} = K''$$

where $c = \frac{\sqrt{m^2+1}+1}{m}$.

The line bisecting the angle between the sides is a special case of this solution. Indeed, the slope of this line is $M \equiv \tan(\frac{1}{2}\tan^{-1}m) = \frac{-1\pm\sqrt{m^2+1}}{m} = -c$ or $-d$, so that $y=Mx$ is a solution for $K''=0$. However, there is also a large family of nonlinear solutions. We have thus proved

<u>Theorem 5.2</u>. A Brooks ribbon with straight sides need not have a straight spine.

*I am indebted to Prof. Quentin Stout for providing this solution.

To gain some intuitive insight into the nature of the nonlinear solutions, consider the case where the sides are perpendicular, say $y=0$ and $x=0$. This is not a special case of our general formulation, since the second side is not of the form $y=g(x)$. However, we can derive the differential equation for this case by the same method; it turns out to be $yy'=x$. (Note that this can be obtained from our general differential equation $myy'+2y=mx$ by dividing through by m and letting $m\to\infty$.) The solution to this equation is simply $y^2=x^2+C$. For $C\neq 0$, this is a family of hyperbolas asymptotic to the line $y=x$, and for $C=0$ we get the line $y=x$ itself. If can be verified (see Figure 19) that if we draw any perpendicular to such a hyperbola, the distances along the perpendicular to the two axes are equal. Note however, that the hyperbola spines do not yield the entire axes as sides. For example, the hyperbola $y^2=x^2-C^2$ shown in Figure 19 cannot generate the interval $[0,2C)$ of the x-axis.

Our straight-sided examples imply

Theorem 5.3. Specifying parts of the sides of a Brooks ribbon does not uniquely determine the spine. //

It should be pointed out that in the straight-sided examples, there is only one linear solution; all the other solutions have higher degree. This suggests the possibility that in general there might be a unique lowest-degree solution.* Unfortunately, this is not so, as we can see from considering the case

*Another possibility is that there might in general be a unique solution of lowest curvature, as there is in the case when the sides are straight; but there seems to be no straightforward way of establishing this.

where one side is a straight line and the other is a parabola, e.g., $y=0$ and $y=ax^2$. Here the intersection points satisfy

$$0 = \frac{x_0-x_1}{h'(x_0)} + h(x_0) \qquad \text{and} \qquad ax_2^2 = \frac{x_0-x_2}{h'(x_0)} + h(x_0)$$

Thus
$$x_1 = x_0+h(x_0)h'(x_0)$$

and
$$x_2 = \frac{-1\pm\sqrt{1+4ah'(x_0)(x_0+h(x_0)h'(x_0))}}{2ah'(x_0)}$$

Substituting in $x_0 = \frac{x_1+x_2}{2}$ gives

$$x_0 = h(x_0)h'(x_0) + \frac{-1\pm\sqrt{1+4ah'(x_0)(x_0+h(x_0)h'(x_0))}}{2ah'(x_0)}$$

so that h satisfies the differential equation

$$x = yy' + \frac{-1\pm\sqrt{1+4ay'(x+yy')}}{2ay'}$$

Transposing and squaring gives

$$\begin{aligned} 1+4ay'(x+yy') &= [1+2ay'(x-yy')]^2 \\ &= 1+4ay'(x-yy')+4a^2y'^2(x-yy')^2 \end{aligned}$$

Thus

$$8ayy'^2 = 4a^2y'^2(x-yy')^2$$

or
$$2y = a(x-yy')^2$$

where we can cancel y'^2 since y is not a constant.

It is not hard to see that this differential equation has no polynomial solution. Note first that it has no linear solution; in fact, if $y=Ax+B$ were a solution, we would have

$$2(Ax+B) = a(x-A(Ax+B))^2$$

This must vanish identically in x; hence the coefficient of each power of x must vanish. Collecting coefficients, we have

$$a(1-A^2)^2x^2 - 2A(1+a(1-A^2)B)x + aA^2B^2-2B \equiv 0$$

In order for the coefficient of x^2 to vanish, we must have $A=\pm 1$; but then the coefficient of x does not vanish, contradiction. Now suppose the equation has a solution of degree exactly $n>1$, say $y=Ax^n+$(terms of lower degree), where $A\neq 0$. Then the coefficient of $x^{2n(n-1)}$ is naA^2, and since this must vanish, we must have $A=0$, contradiction. In summary, we have proved

Proposition 5.4. If one side of a Brooks ribbon is a straight line and the other is a parabola, the spine is not a polynomial. |

It would be useful to obtain explicit solutions for the spine when the sides are polynomials of low degree, but the differential equation of the spine is extremely complicated when both sides are nonstraight -- e.g., even when they are both circular arcs.

6. The equal angle case ("Brady ribbons")

Finally, we consider L-ribbons satisfying the condition that the generator always makes equal angles with the sides of the ribbon. We call such ribbons "Brady ribbons."

Note first that it is not obvious how to generate a Brady ribbon from an arbitrary given spine. Let the equation of the spine be $y=h(x)$, and let the generator centered at point (x_0,y_0) of the spine have half-length r_0 and slope $\tan\theta_0$. Then the endpoints of the generator are at $(x_0 \pm r_0\cos\theta_0,$ $y_0 \pm r_0\sin\theta_0)$, where $y_0=h(x_0)$. The loci of the endpoints are the sides of the ribbon; thus the slopes of the sides at the endpoints are

$$\frac{d(y_0 \pm r_0\sin\theta_0)}{d(x_0 \pm r_0\cos\theta_0)} = \frac{y_0' \pm r_0'\sin\theta_0 \pm r_0\theta_0'\cos\theta_0}{1 \pm r_0'\cos\theta \mp r_0\theta_0'\sin\theta_0}$$

If we call these slopes $\tan\theta_1$ and $\tan\theta_2$, respectively, then the equal-angle condition means that we must have $\frac{\theta_1+\theta_2}{2} = \theta_0$. In principle, we can solve this equation to find pairs of functions r_0 and θ_0 that generate Brady ribbons. In practice, it may be hard to solve the equations in general: but in any case, Brady's definition was not intended to be generative.

The situation is somewhat better as regards recovery. To begin with, we have

Proposition 6.1. If a Brady ribbon has parallel straight sides, its spine must be a segment of the straight line parallel to the sides and midway between them, but its generators can make arbitrary angles with the spine.

Proof: Any line that intersects two parallel lines makes equal angles with them, and its midpoint is halfway between them. As we saw in Figure 13, there are many ways of defining the generators so that no two of them intersect. ||

Proposition 6.2. If a Brady ribbon has nonparallel straight sides, its spine must be a segment of the angle bisector of the sides, and its generators must be perpendicular to the spine.

Proof: Let the sides have slopes $\tan\theta_1$ and $\tan\theta_2$, where $\theta_1 \neq \theta_2$. An arbitrary line of slope $\tan\theta$ makes angles $\theta-\theta_1$ and $\theta_2-\theta$ with the sides. Thus there is only one slope for which these angles are equal, namely $\tan\theta$ where $\theta=\frac{\theta_1+\theta_2}{2}$. Thus all generators must be parallel, and evidently they are perpendicular to the angle bisector. ||

Propositions 6.1 and 6.2 show that if a Brady ribbon has straight sides, its axis is uniquely determined, but its generators are not uniquely determined if the sides are parallel. Note that the situation here is exactly the reverse of that for Brooks ribbons with straight sides; if the sides are parallel, there is only one way of generating a Brooks ribbon (Proposition 5.1), but if they are nonparallel there are many ways (Theorem 5.2).

Proposition 6.2 also holds if just one side is straight. In fact, we have

Theorem 6.3. If a Brady ribbon has just one straight side, its spine and generators are uniquely determined.

Proof: Let the straight side have slope $\tan\theta_1$, let P be any point on the other side, and let the tangent at P have slope θ_2, where $\theta_2 \neq \theta_1$. Just as in the proof of Proposition 6.2, there is a unique line through P that makes equal angles with the straight side and with the tangent at P -- namely, the line having slope $\tan\theta$, where $\theta = \frac{\theta_1+\theta_2}{2}$. Thus at every P for which $\theta_2 \neq \theta_1$, the generator is uniquely determined. Moreover, at any P for which $\theta_2 = \theta_1$, we must take the generator perpendicular to the two sides in order to insure continuity of its slope. Thus all the generators are uniquely determined, and the spine is the locus of their midpoints. |

For arbitrarily shaped sides s and t, let $P \in s$. If there exists $Q \in t$ such that the normals to s at P and to s' at Q are parallel, then the line segment $\overline{PQ}$ is a generator, since it makes equal angles with the normals; and if there exists more than one such Q, there is more than one generator with endpoint P. On the other hand, as Figure 20 shows, the normals at P and Q need not be parallel in order for PQ to be a generator. Thus we cannot say anything simple about uniqueness in the general case. Note that if s and t are circular arcs, it is easy to construct parallel pairs of normals. We simply draw the line AB joining the centers of the circles (see Figure 21); if the normal at $P \in s$ makes angle ϕ with AB, we let Q be the point of t at which the normal makes the same angle with AB, and PQ is then a generator, since the normals at P and Q are parallel. If the circles are facing each another, this is not a useful construction, since the PQ's must cross one another, as illustrated in Figure 21a. However, if the circles are facing the same way, the construction yields a non-selfintersecting Brady ribbon, as illustrated in Figure 21b, and as also discussed by Brady.

A variant of the construction in Figure 21b can be used to show that thick Brady ribbons can make sharp turns. In Figure 21b, if t is very tiny, the generators PQ will have approximately the same length; hence the locus of their centers, i.e., the spine, is approximately a circular arc parallel to s and with about half its radius. Thus the ribbon is more than twice as thick as the radius of curvature of its spine, but it is still able to turn without intersecting itself. This example also shows that there exist Brady ribbons that are not Brooks ribbons.

Finally, we show that every Blum ribbon is a Brady ribbon. Indeed, referring to the last paragraph of Section 3, triangle OPP' is isosceles; hence PP' makes equal angles with the normals OP and OP' to the sides of the ribbon. Thus PP' is a Brady generator, and the locus of its midpoint is a Brady spine. (Note that this spine is not necessarily the same as the Blum spine, which is the locus of O.) If R is a Blum ribbon, we can recover its Blum generators as in Section 3, giving us the pairs (P,P'); this then gives us a set of Brady generators for R, namely the segments PP'. (There is no guarantee, however, that this set is unique.)

7. Some addenda and special cases

In the last two sections we have ignored what happens at the ends of a ribbon. By definition, the ends of a Blum ribbon must be circular arcs, while those of a Brooks or Brady ribbon must be straight. In the case of L-ribbons (i.e., ribbons having line segment generators) we can "shape" the ends by attaching to each end what we shall call a "generalized sector." This is simply a pencil of ray segments emanating from each endpoint of the segment over a 180° sector (in the half-place bounded by the last generator of the ribbon and not containing the adjacent generators). The lengths of the ray segments can vary in any desired way. In particular, if we make the segments all equal, the generalized sector becomes a semidisk, so that the ends of the ribbon are rounded. Another way of shaping the ends of an L-ribbon is to attach to each end a "generalized wedge"; this is a pencil of ray segments emanating from the endpoint of one of the sides, and covering an angular sector bounded on one side by the last generator of the ribbon. Generalized wedges would be a natural way of completing ribbons such as that shown in Figure 16. Note that in both generalized sectors and generalized wedges, the generating ray segments all have one endpoint in common, rather than being disjoint as in the ribbon case.

If we do not ignore what happens at the ends, our three special classes of ribbons (Blum, Brooks, and Brady) are all incomparable. Blum ribbons have rounded ends, while Brooks and Brady ribbons have flat ends. There exist Brady ribbons that are not Brooks ribbons, as we saw at the end of Section 6. Conversely, a Brooks ribbon such as that in Figure 19 is not a Brady ribbon; Brady's method can generate straight-sided ribbons, but the sides must be symmetric around their angle bisector.

Even if we ignore the ends, there are many types of Brooks or Brady ribbons that are not Blum ribbons -- e.g., Figure 10a; and there are Brady ribbons that are not Brooks or Blum ribbons, as we saw at the end of Section 6. The remaining questions are: Ignoring the ends, is every Blum ribbon a Brooks ribbon? Is every Brooks ribbon a Brady ribbon? We will not settle these questions here in general, but we will settle them in the case where the spine is straight.

Proposition 7.1. If the spine is straight, and we ignore the ends, every Blum ribbon is a Brooks ribbon.

Proof: Let R be a Blum ribbon whose spine lies along the x axis, and let O be any point on the spine. Since R is a union of disks centered on the x-axis, it is clear that the vertical line through O can only meet R in a single connected segment. Let P and Q be the border points of R directly above and below Q. Any disk centered on the x axis that contains P also contains Q, and vice versa. Hence the (unique) generator of R that touches b_R at P also touches it at Q, so that P and Q are equidistant from the x axis. At each point O of the spine, $\overline{PQ}$ is a Brooks generator, since it is perpendicular to the spine and O is its midpoint. The Brooks ribbon generated by this set of generators is evidently R (except at the ends).

Proposition 7.2. If the spine is straight, every Brooks ribbon is a Brady ribbon.

Proof: Let R be a Brooks ribbon with a straight spine, say lying along the x axis. Let $G(x)$ be the generator of R at x (so that $G(x)$ is a vertical line segment with x as its midpoint), and let $r(x)$ be the half-length of $G(x)$. The slopes of the sides of R at the endpoints of $G(x)$ are evidently $\pm r'(x)$; hence the sides make equal angles with $G(x)$, so that R is a Brady ribbon. |

Even if the spine is straight, a Brady ribbon need not be a Brooks ribbon; see Figure 10b. Similarly, a Brooks ribbon need not be a Blum ribbon (Figure 10a). Thus for straight spines, *ignoring the ends, the three classes of ribbons are strictly* nested: Blum $\subsetneq$ Brooks $\subsetneq$ Brady.

8. Concluding remarks

We have discussed at some length various ways of defining "generalized ribbons." In particular, we have considered three specific models, due to Blum, Brooks, and Brady, respectively. Blum's model seems to have the least generative capacity, but it has unique recoverability. Blum ribbons are limited in their flexibility (e.g., in terms of turn radius), but they are also limited in their ability to generate non-ribbon like shapes. Brady's model has somewhat more generative capacity than Brooks', and its recoverability properties also seem to be better. Its main disadvantage is that generation is not a straightforward process; it is not easy to specify how to define the set of generators so as to insure that they satisfy Brady's equal angle condition.

It would be of interest to generalize the results of this paper to three dimensions by defining and comparing various classes of generalized cylinders (or cones). In 3D the spine is a space curve (or rather arc). The 3D analog of Blum's model uses a ball (i.e., a solid sphere) as generator, while the analogs of the 2D schemes based on line segment generators use a planar figure (such as a disk!) as generator. Here we can consider a Brooks-like restriction in which the disk is required to remain at a fixed angle to the spine; it is less obvious how to define a 3D analog of Brady's equal-angles restriction. Note that if we use a line segment as generator in 3D, and allow its length and spatial orientation to vary, we obtain generalized "space ribbons" rather than cylinders or cones.

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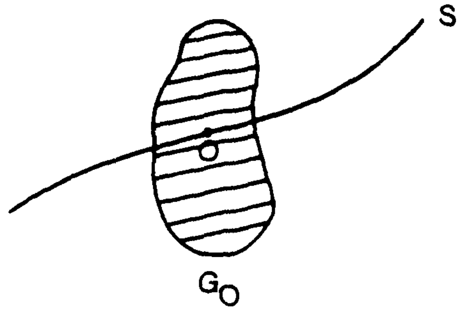


Figure 1. The generation process.

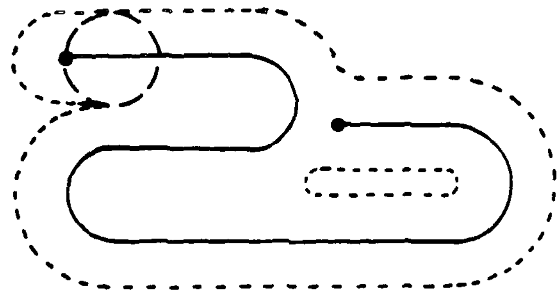


Figure 2. If R were allowed to intersect itself we could generate very non-ribbonlike shapes.

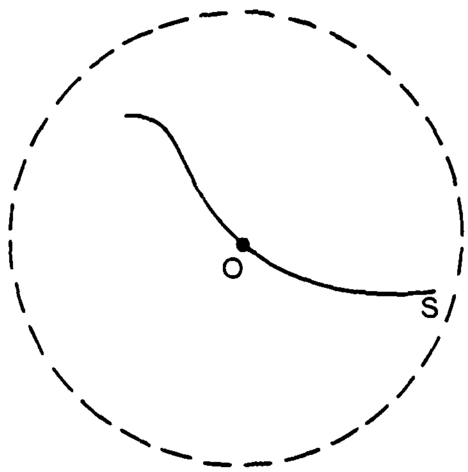


Figure 3. If G's could contain one another, the shape of R might not be strongly influenced by that of S. Here G_O is large, and the G's get rapidly smaller as we move away from O, so that G_O is all of R.

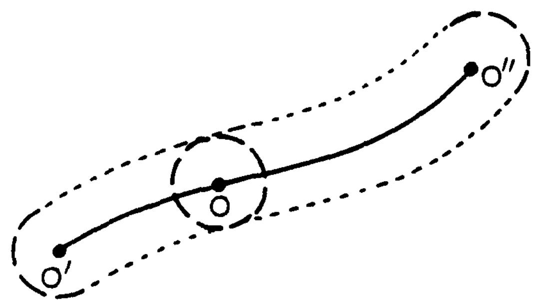


Figure 4. The sides (dotted) and ends (dashed) of R.

(a)

(b)

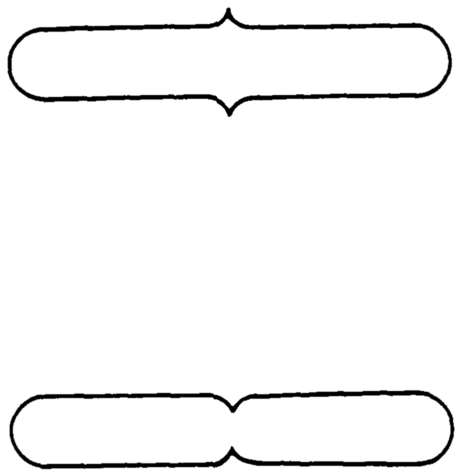


Figure 5. A thick Blum ribbon can have points of high negative curvature on its border (b), but not points of high positive curvature (a).

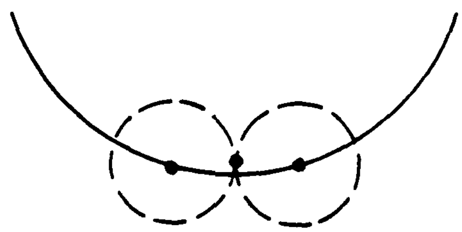


Figure 6. When the spine is curved, disks that just touch do not touch on the spine.

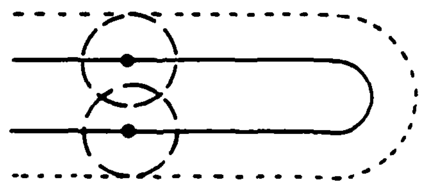


Figure 7. This example violates non selfintersection and also nonmaximality.

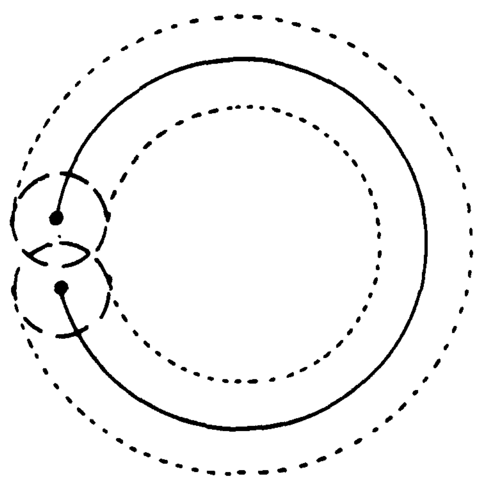


Figure 8. This example violates nonselfintersection but not nonmaximality.

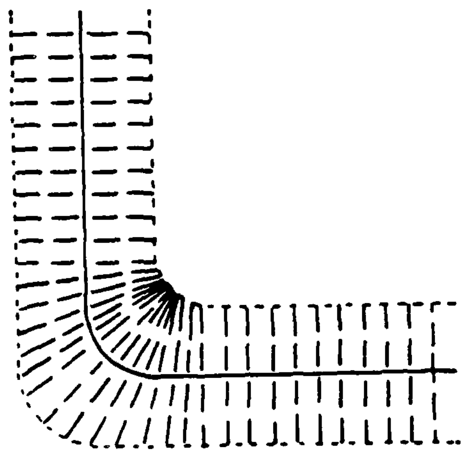
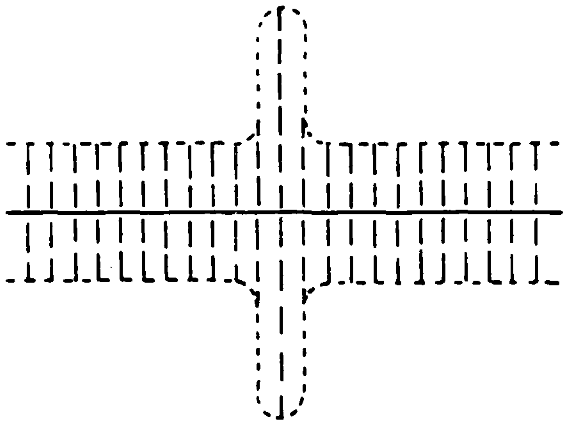


Figure 9. Thick L-ribbons can make sharp turns.

(a)



(b)

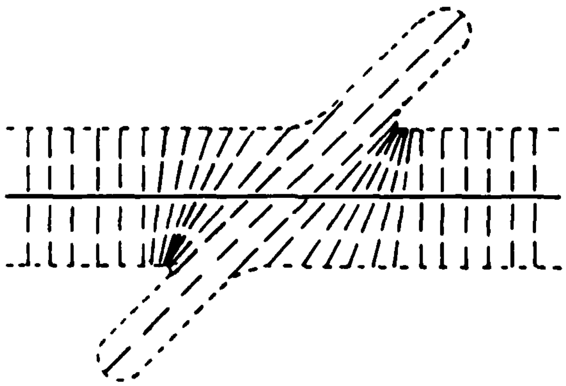


Figure 10. L-ribbons can have long protuberances on their borders (a,b), but not if they look like (c,d,e).

(c)

(d)

(e)

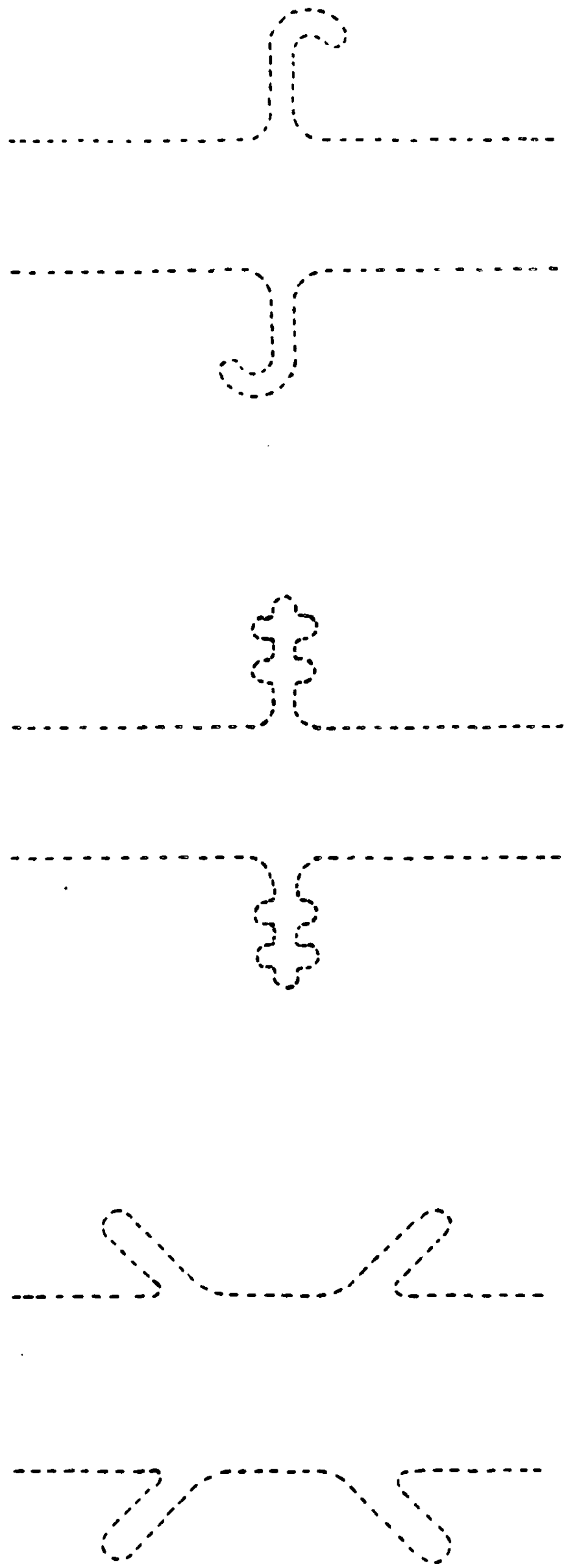


Figure 10 (continued).

(a)

(b)

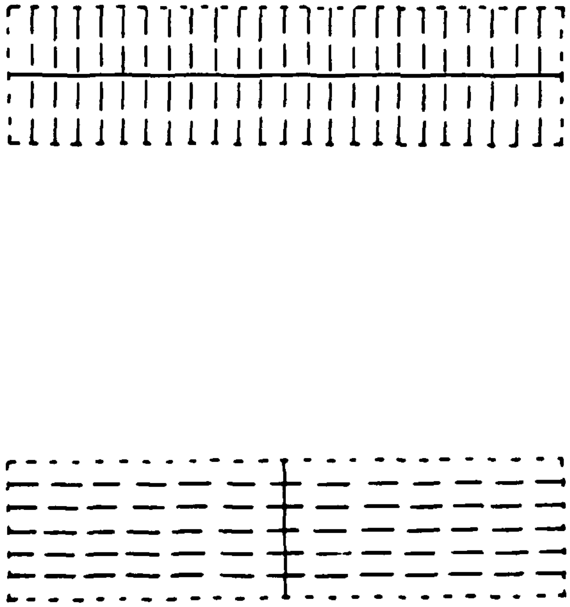


Figure 11. We prefer (a) over (b) because it is more elongated.

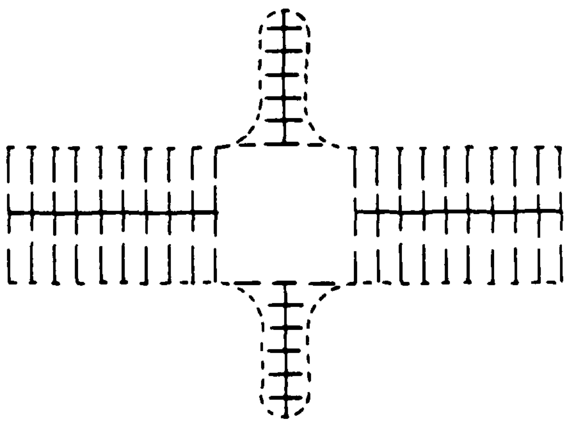
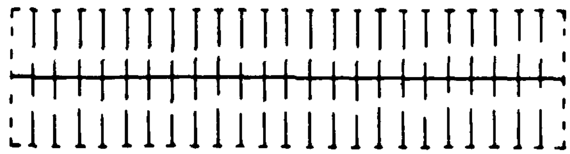


Figure 12. The protuberances have much greater elongatedness with respect to spines of their own.

(a)



(b)

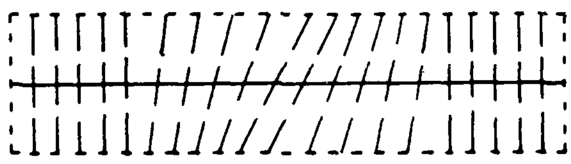


Figure 13. L-ribbons are highly ambiguous; a given *shape can be generated* in many different ways.

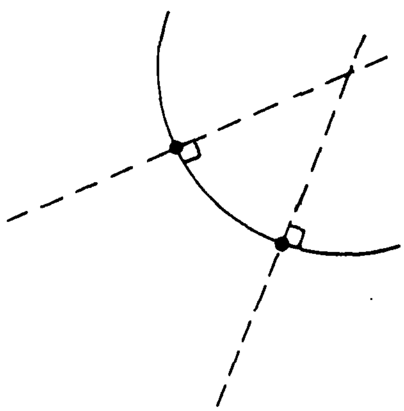


Figure 14. The thickness of a Brooks ribbon cannot exceed twice the radius of curvature of its spine.

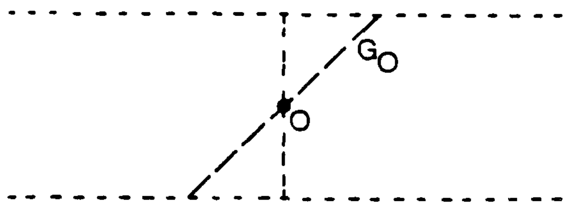


Figure 15. If a Brooks ribbon has parallel straight sides, its spine must be parallel to them and midway between them.

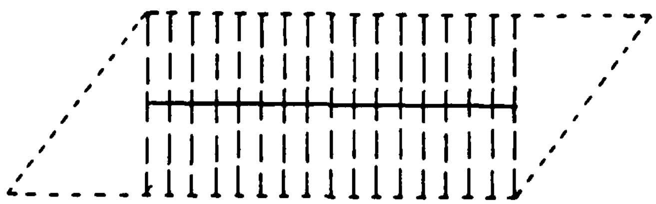


Figure 16. A parallelogram is not a Brooks ribbon.

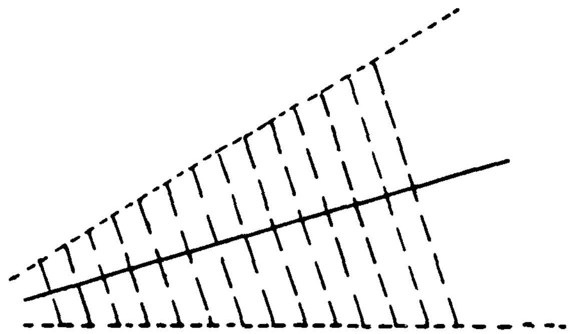


Figure 17. A Brooks ribbon with straight sides.

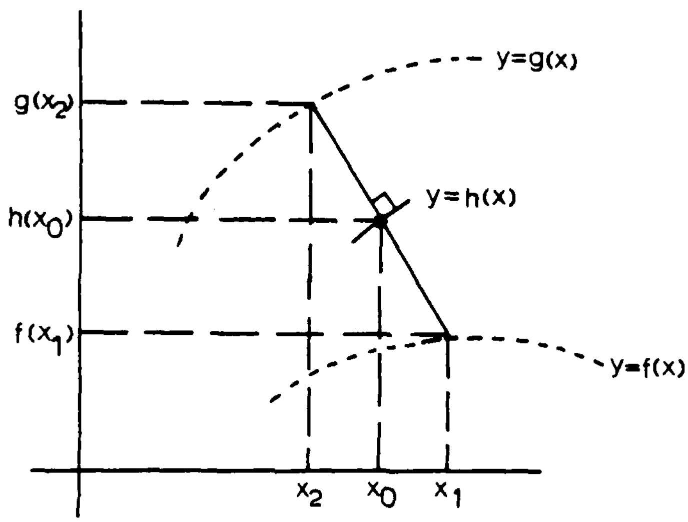


Figure 18. Deriving the differential equation of the spine y=h(x).

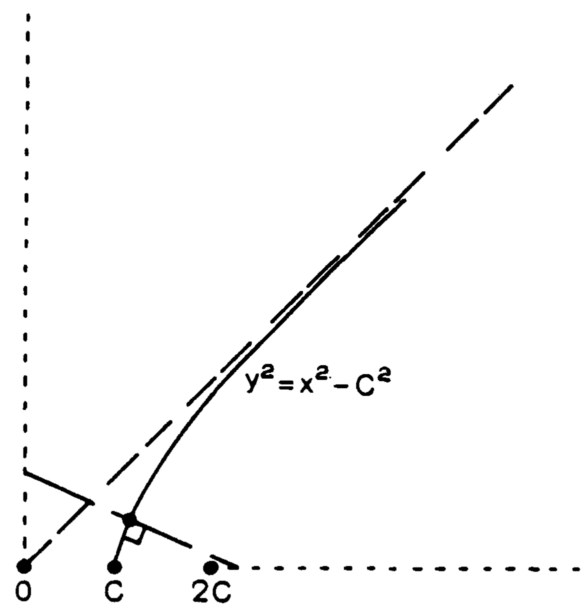


Figure 19. A Brooks ribbon with straight sides need not have a straight spine.

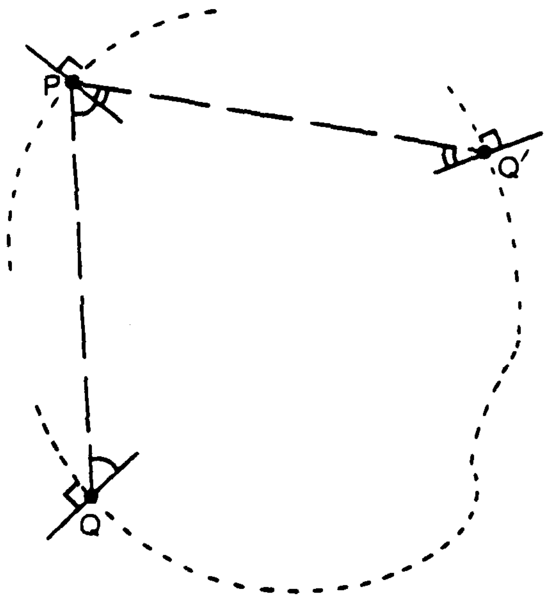
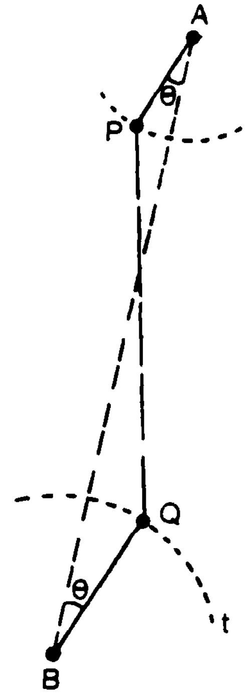


Figure 20. Brady generators don't require parallel normals; PQ' and PQ are both generators.

(a)



(b)

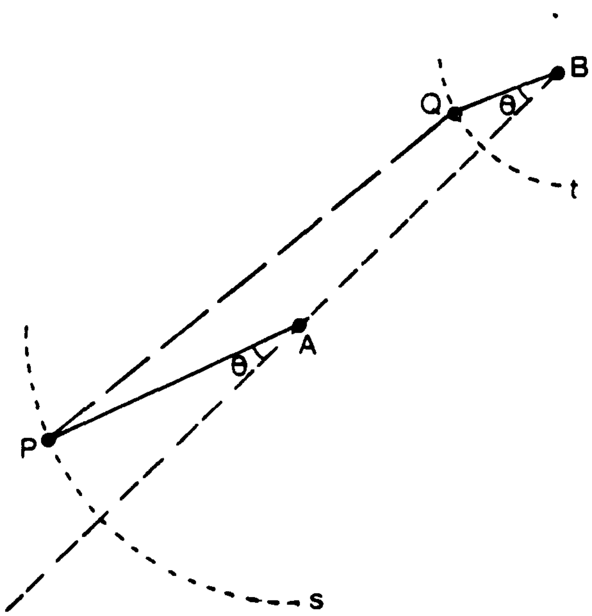


Figure 21. Pairs of circular arcs sometimes define Brady ribbons.

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Classes of "ribbonlike" planar shapes can be defined by specifying an arc, called the spine or axis, and a geometric figure such as a disk or line segment, called the generator, that "sweeps out" the shape by moving along the spine, changing size as it moves. Shape descriptions of this type have been considered by Blum, Brooks, Brady, and others. This paper considers such descriptions from the standpoints of both generation and recovery (i.e., given a shape generated in this way, to determine the axis and generation rule that gave rise to it), and discusses their relative advantages and disadvantages.

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